International Journal of Theoretical Physics, Vol. 22, No. 10, 1983

Transformations between Inertial and Linearly Accelerated Frames of Reference

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Received January 7, 1981

Transformation equations between inertial and linearly accelerated frames of reference are derived and these transformation equations are shown to be compatible, where applicable, with those of special relativity. The physical nature of an accelerated frame of reference is unambiguously defined by means of an equation which relates the velocity of all points within the accelerated frame of reference to measurements made in an inertial frame of reference.

1. INTRODUCTION

Many transformation equations between inertial frames of reference and accelerated frames of reference have been proposed to date (see, e.g., Lass, 1963; Marsh, 1965; Atwater, 1974) but all have been based upon *ad hoc* assumptions and there is no experimental evidence to verify that any of the suggested transformations would predict the results of observations made in accelerated frames.

The present paper considers the case of accelerated systems in which the material energy tensor is everywhere zero, i.e., systems of negligible mass. Coordinate transformation equations between an inertial frame of reference and a linearly accelerated frame of reference are derived and the differences between *coordinate measurements* and *physical measurements* are fundamental to this derivation. As pointed out by Marsh (1965) the spatial and temporal coordinates of an accelerated frame of reference do not necessarily correspond to physical measurements made at points other than the origin of the accelerated system. If physical measurements are to be made at points other than the origin then there must be observers at these other points. The physical measurement of elements of space and time made by each of these observers within his own infinitesimally small neighborhood must be related by the Lorentz transformations to measurements made in the inertial frame. Marsh (1965) gives as an example the equation of motion of the origin of the inertial system in terms of physical measurements made in the accelerated system. In order to obtain such an equation it would be necessary for an infinite number of observers to be situated in the accelerated system and for these observers to make measurements of the motion of the inertial origin if it entered their own infinitesimally small neighborhoods. The collected information of this set of observers would then give the equation of motion of the inertial origin in terms of physical measurements. On the other hand, the motion of the inertial origin according to an observer who remains at the origin of the accelerated system would be given by the relevant equations in terms of the coordinates of the accelerated system.

It is a well known result of relativity theory that it is impossible to accelerate a perfectly rigid body. It is therefore to be expected that all points in an accelerated frame of reference which is moving parallel to the x axis of an inertial frame, $\Sigma(x, y, z, t)$, will have instantaneous velocities, as measured in the inertial frame, which vary according to the magnitude of their x' coordinate within the accelerated frame $\Sigma'(x', y', z', t')$. The magnitude of the velocity of such a point, according to an observer in the inertial frame, we shall denote by $Z(x')$.

2. DERIVATION OF THE COORDINATE TRANSFORMATION EQUATIONS

Throughout the paper we shall denote the inertial frame of the laboratory by $\Sigma(x, y, z, t)$ and the accelerating frame by $\Sigma'(x', y', z', t')$. We shall assume that the origin of Σ' is accelerating with an acceleration $a = dv/dt = d^2x/dt^2$ with respect to Σ. The force producing this acceleration, as measured in Σ, will not, in general, be constant with time. The origins of Σ and Σ' are assumed to coincide at time $t = t' = 0$ and v is the velocity of the *origin* of Σ' as measured in Σ and is in a direction parallel to both the x and the x' axes. v will be a function of time.

Let $d\sigma'$ be the *physically* measured local spatial line element according to an observer in Σ'. Hence, since local geometry is Euclidean,

$$d\sigma'^2 = d\sigma_x'^2 + d\sigma_y'^2 + d\sigma_z'^2 \tag{1}$$

where $d\sigma_x'$, $d\sigma_y'$, and $d\sigma_z'$ are the physically measured local spatial line elements in the x', y', and z' directions, respectively. The physically measured local time element in Σ' we shall denote by dT'. Since these physically measured local elements of space and time must be related by the Lorentz transformations to measurements made in the inertial frame, we can write

$$d\sigma_x' = (dx - Z\,dt)(1 - Z^2/c^2)^{-1/2} \tag{2}$$

$$d\sigma_y' = dy \tag{3}$$

$$d\sigma_z' = dz \tag{4}$$

$$dT' = (dt - Z\,dx/c^2)(1 - Z^2/c^2)^{-1/2} \tag{5}$$

where Z is the velocity of the observer in Σ' according to measurements made in Σ. Equations (2)–(5) must hold for local physical measurements made by observers situated anywhere in Σ'. Hence, since the accelerated frame cannot be perfectly rigid, Z will be a function of x'. *The physically measured local elements of space and time*, $d\sigma_x'$, $d\sigma_y'$, $d\sigma_z'$, *and* dT', *as given in equations* (2)–(5) *must be uniquely related to the coordinate elements of space and time* dx', dy', dz', *and* dt' *but will be identical to them only at the origin*, $x' = 0$. In order to find these unique relationships we shall firstly assume that since the y, z and y', z' coordinates are perpendicular to the direction of motion they will be related, as in special relativity, by

$$y' = y \tag{6}$$

and

$$z' = z \tag{7}$$

thus giving, by equations (3) and (4),

$$d\sigma_y' = dy = dy' \tag{8}$$

and

$$d\sigma_z' = dz = dz' \tag{9}$$

The third spatial line element, $d\sigma_x'$, will be related to the coordinate element, dx', by the relationship

$$d\sigma_x' = h(x', t')\,dx' \tag{10}$$

where h is a function (to be determined) of the coordinates x', t'. dt' does not appear in equation (10) because $d\sigma'$ is the *spatial* part of the line element (see, e.g., Atwater, 1974). The proper time element, dT', will be related to the coordinate elements of space and time by the relationship

$$dT' = f(x', t')\,dt' + g(x', t')\,dx' \tag{11}$$

where f and g are functions (to be determined) of the coordinates x', t'.

In Σ the flat space-time of the Minkowski metric is given by

$$ds^2 = c^2\,dt^2 - dx^2 - dy^2 - dz^2 \tag{12}$$

Using equations (2)–(5) enables equation (12) to be written in the form

$$ds^2 = c^2\,dT'^2 - d\sigma_x'^2 - d\sigma_y'^2 - d\sigma_z'^2 \tag{13}$$

which, together with equations (8)–(11), enables us to write the metric of the linearly accelerated system Σ' as

$$ds^2 = c^2 f^2\,dt'^2 + 2c^2 fg\,dx'\,dt' - (h^2 - c^2 g^2)\,dx'^2 - dy'^2 - dz'^2 \tag{14}$$

from which the spatial line element can be confirmed to be given by

$$d\sigma'^2 = h^2\,dx'^2 + dy'^2 + dz'^2 \tag{15}$$

in accord with equation (1) and equations (8)–(10). From equations (2) and (10)

$$dx' = (dx - Z\,dt)\left(1 - \frac{Z^2}{c^2}\right)^{-1/2} h^{-1} \tag{16}$$

and from equations (5) and (11)

$$dt' = \left[(h + gZ)\,dt - (Zh + gc^2)c^{-2}\,dx\right]\left(1 - \frac{Z^2}{c^2}\right)^{-1/2} f^{-1}h^{-1} \tag{17}$$

Since the whole of the accelerated frame of reference is described by Σ', equations (16) and (17) must be exact differentials. Hence,

$$\frac{\partial x'}{\partial t} = -Z\left(1-\frac{Z^2}{c^2}\right)^{-1/2} h^{-1}, \qquad \frac{\partial x'}{\partial x} = \left(1-\frac{Z^2}{c^2}\right)^{-1/2} h^{-1}$$

$$\frac{\partial t'}{\partial t} = (h+gZ)\left(1-\frac{Z^2}{c^2}\right)^{-1/2} f^{-1}h^{-1}, \tag{18}$$

$$\frac{\partial t'}{\partial x} = -(Zh+gc^2)\left(1-\frac{Z^2}{c^2}\right)^{-1/2} c^{-2}f^{-1}h^{-1}$$

According to an observer in Σ the origin of Σ' will have an equation of motion given by $x=\int_0^t v\,dt$; therefore, assuming a linear (in x) relationship between x' and x we can write

$$x' = k(t)\left(x - \int_0^t v\,dt\right) \tag{19}$$

where k is a function of t only. Differentiating equation (19) gives

$$dx' = \frac{dk}{dt}x'k^{-1}\,dt + k(dx - v\,dt) \tag{20}$$

Therefore, when $x'=0$, i.e., at the origin of Σ',

$$dx' = k(dx - v\,dt) \tag{21}$$

But, we have already stated that the Lorentz transformations must hold for the observer at the origin of the accelerated frame of reference *even when coordinate measurements are used*. Therefore,

$$dx' = (dx - v\,dt)(1 - v^2/c^2)^{-1/2} \tag{22}$$

when $x'=0$, where v is the instantaneous velocity of the origin of Σ' according to an observer in Σ. Comparing equations (21) and (22) gives

$$k = (1 - v^2/c^2)^{-1/2} \tag{23}$$

From equations (18), (20), and (23) we can see that

$$h = (1 - v^2/c^2)^{1/2}(1 - Z^2/c^2)^{-1/2} \tag{24}$$

and

$$Z = v(1 - Wx') \tag{25}$$

where

$$W = ac^{-2}(1 - v^2/c^2)^{-1/2} \tag{26}$$

Now, assuming a linear relationship between t' and x we can write

$$t' = j(t)x + m(t) \tag{27}$$

where j is a function of t only. Using equation (19) to substitute for x into equation (27) gives

$$t' = jx'(1 - v^2/c^2)^{1/2} + m + j\int_0^t v\,dt \tag{28}$$

At the origin of Σ', i.e., when $x' = 0$, equation (28) becomes

$$t'_{x'=0} = m + j\int_0^t v\,dt \tag{29}$$

If coordinate time and proper time are to be identical at the origin of the Σ' coordinate system then,

$$t'_{x'=0} = \int_0^t (1 - v^2/c^2)^{1/2}\,dt \tag{30}$$

and comparing equations (29) and (30) gives

$$m(t) = \int_0^t (1 - v^2/c^2)^{1/2}\,dt - j\int_0^t v\,dt \tag{31}$$

which, when substituted into equation (27), gives

$$t' = jx + \int_0^t (1 - v^2/c^2)^{1/2}\,dt - j\int_0^t v\,dt \tag{32}$$

Differentiating equation (32) and using equations (19) and (23) gives

$$dt' = x'\left(1 - \frac{v^2}{c^2}\right)^{1/2}\frac{dj}{dt}\,dt + j\,dx + \left(1 - \frac{v^2}{c^2}\right)^{1/2}dt - jv\,dt \tag{33}$$

Therefore, when $x' = 0$,

$$dt' = j\,dx + \left[\left(1 - v^2/c^2\right)^{1/2} - jv\right] dt \tag{34}$$

But, the Lorentz transformations must hold for the observer at the origin of the accelerated frame of reference *even when coordinate measurements are used*. Therefore

$$dt' = \left(dt - v\,dx/c^2\right)\left(1 - v^2/c^2\right)^{-1/2} \tag{35}$$

when $x' = 0$. Comparing equations (34) and (35) gives

$$j(t) = -vc^{-2}\left(1 - v^2/c^2\right)^{-1/2} \tag{36}$$

and equation (33) becomes

$$dt' = \left(c^2 Z\,dt - v^2\,dx\right)v^{-1}c^{-2}\left(1 - v^2/c^2\right)^{-1/2} \tag{37}$$

From equations (18), (24), and (37) we find that

$$f = v\left(1 - Z^2/c^2\right)^{1/2} Z^{-1}\left(1 - v^2/c^2\right)^{-1/2} \tag{38}$$

and

$$g = -Z\left(1 - v^2/Z^2\right)c^{-2}\left(1 - Z^2/c^2\right)^{-1/2}\left(1 - v^2/c^2\right)^{-1/2} \tag{39}$$

From equations (6), (7), (19), (23), (32), and (36) the transformation equations between an inertial frame of reference and a linearly accelerated frame of reference are

$$\begin{aligned} x' &= \left(x - \int_0^t v\,dt\right)\left(1 - v^2/c^2\right)^{-1/2} \\ y' &= y \\ z' &= z \\ t' &= \int_0^t \left(1 - v^2/c^2\right)^{1/2} dt - vc^{-2}\left(1 - v^2/c^2\right)^{-1/2}\left(x - \int_0^t v\,dt\right) \end{aligned} \tag{40}$$

which reduce to give the transformation equations of special relativity when

v is a constant. Differentiating equations (40) we obtain

$$\begin{aligned} dx' &= (dx - Z\,dt)(1 - v^2/c^2)^{-1/2} \\ dy' &= dy \\ dz' &= dz \\ dt' &= (Zc^2\,dt - v^2\,dx)(1 - v^2/c^2)^{-1/2} v^{-1} c^{-2} \end{aligned} \tag{41}$$

and

$$\begin{aligned} dx &= (dx' + v\,dt')(1 - v^2/c^2)^{-1/2} \\ dy &= dy' \\ dz &= dz' \\ dt &= (dt' + v\,dx'/c^2)(1 - v^2/c^2)^{-1/2} v Z^{-1} \end{aligned} \tag{42}$$

in which

$$Z = v(1 - Wx') \tag{43}$$

and

$$W = ac^{-2}(1 - v^2/c^2)^{-1/2} \tag{44}$$

Hence, the metric of the linearly accelerated system Σ' is given by

$$\begin{aligned} ds^2 = {} & c^2 v^2 Z^{-2}(1 - Z^2/c^2)(1 - v^2/c^2)^{-1} dt'^2 \\ & - 2v(1 - v^2/Z^2)(1 - v^2/c^2)^{-1} dx'\,dt' \\ & - (1 - v^4/c^2 Z^2)(1 - v^2/c^2)^{-1} dx'^2 \\ & - dy'^2 - dz'^2 \end{aligned} \tag{45}$$

and the spatial line element is

$$d\sigma'^2 = (1 - v^2/c^2)(1 - Z^2/c^2)^{-1} dx'^2 + dy'^2 + dz'^2 \tag{46}$$

The above transformation equations can now be used to evaluate the transformation equations, between Σ and Σ', of velocity and acceleration.

3. TRANSFORMATIONS OF VELOCITY AND ACCELERATION

If we let the components of the velocity of a moving body with respect to the inertial frame, Σ, be given by

$$q_x = \frac{dx}{dt}, \qquad q_y = \frac{dy}{dt}, \qquad q_z = \frac{dz}{dt} \tag{47}$$

and the components of velocity of the same body with respect to the accelerated system, Σ', be given by

$$q'_x = \frac{dx'}{dt'}, \qquad q'_y = \frac{dy'}{dt'}, \qquad q'_z = \frac{dz'}{dt'} \tag{48}$$

then, from equation (42) we have that

$$q_x = (q'_x + v)Zv^{-1}(1 + vc^{-2}q'_x)^{-1} \tag{49}$$

$$q_y = q'_y(1 - v^2c^{-2})^{1/2}Zv^{-1}(1 + vc^{-2}q'_x)^{-1} \tag{50}$$

and the z transformation can be obtained by replacing y by z in equation (50). Setting $a = 0$ in equations (49) and (50) we obtain the velocity transformation equations of special relativity.

Similarly, if the components of acceleration of a moving body with respect to Σ are given by

$$a_x = \frac{d^2x}{dt^2}, \qquad a_y = \frac{d^2y}{dt^2}, \qquad a_z = \frac{d^2z}{dt^2} \tag{51}$$

and the components of acceleration of the same body with respect to Σ' are given by

$$a'_x = \frac{d^2x'}{dt'^2}, \qquad a'_y = \frac{d^2y'}{dt'^2}, \qquad a'_z = \frac{d^2z'}{dt'^2} \tag{52}$$

then, by differentiating equations (49) and (50) we obtain

$$a'_x = \frac{(1-v^2/c^2)^{1/2}}{(1-Wx'-vq_x/c^2)^3}\left\{a_x(1-Wx')\left(1-\frac{v^2}{c^2}\right)+x'q_x\left(1-\frac{v^2}{c^2}\right)\frac{dW}{dt}\right.$$

$$\left.+a\left[\frac{2q_x^2}{c^2}-\frac{(1-Wx')q_xv}{c^2}-(1-Wx')^2\right]\right\} \tag{53}$$

$$a'_y = \frac{(1-v^2/c^2)}{(1-Wx'-vq_x/c^2)^3}\left\{a_y\left(1-Wx'-\frac{vq_x}{c^2}\right)+\frac{q_ya_xv}{c^2}+q_yx'\frac{dW}{dt}\right.$$

$$\left.+2q_y[q_x-v(1-Wx')]a\left(1-\frac{v^2}{c^2}\right)^{-1}c^{-2}\right\} \tag{54}$$

and the z transformation can be obtained by replacing y by z in equation (54). Setting $a=0$ in equations (53) and (54) we obtain the acceleration transformation equations of special relativity.

The above velocity and acceleration transformation equations are applicable to transformations between *physical* measurements of velocity and acceleration in Σ and *coordinate* measurements of velocity and acceleration in Σ'. Transformation equations between *physical* measurements of velocity and acceleration in Σ and *physical* measurements of velocity and acceleration in Σ' are obtained by denoting

$$Q'_x = \frac{d\sigma'_x}{dT'}, \qquad Q'_y = \frac{d\sigma'_y}{dT'}, \qquad Q'_z = \frac{d\sigma'_z}{dT'} \tag{55}$$

as the components of velocity as physically measured in Σ' and

$$A'_x = \frac{d^2\sigma'_x}{dT'^2}, \qquad A'_y = \frac{d^2\sigma'_y}{dT'^2}, \qquad A'_z = \frac{d^2\sigma'_z}{dT'^2} \tag{56}$$

as the components of acceleration as physically measured in Σ'. Thus,

$$Q'_x = (q_x - Z)(1 - Zq_x/c^2)^{-1} \tag{57}$$

$$Q'_y = q_y(1 - Z^2/c^2)^{1/2}(1 - Zq_x/c^2)^{-1} \tag{58}$$

and the z transformation can be obtained by replacing y by z in equation (58). Equations (57) and (58) are identical in form to the velocity transformation equations of special relativity when Z is replaced by v. The velocity of a point at rest in the inertial system Σ, according to physical measurements made in Σ', is given by setting $q_x = q_y = q_z = 0$ in equations (57) and (58), giving

$$v_I' = Q_x' = -Z, \qquad Q_y' = 0, \qquad Q_z' = 0 \tag{59}$$

where v_I' is the velocity of Σ according to physical measurements made in Σ'. The velocity v_a of Σ' according to physical measurements made in Σ is found by setting $Q_x' = Q_y' = Q_z' = 0$ in equations (57) and (58), giving

$$v_a = q_x = Z, \qquad q_y = 0, \qquad q_z = 0 \tag{60}$$

thus giving a symmetrical velocity relationship between the accelerated frame and the inertial frame. Similarly,

$$A_x' = \left[a_x(1 - Z^2/c^2) - (1 - q_x^2/c^2)\, dZ/dt\right](1 - Z^2/c^2)^{1/2}(1 - Zq_x/c^2)^{-3} \tag{61}$$

$$A_y' = \left[a_y(1 - Zq_x/c^2) + q_y Z a_x/c^2 + q_y(q_x - Z)c^{-2}(1 - Z^2/c^2)^{-1} dZ/dt\right](1 - Z^2/c^2)(1 - Zq_x/c^2)^{-3} \tag{62}$$

and the z transformation can be obtained by replacing y by z in equation (62). The acceleration of a point at rest in the inertial system Σ, according to physical measurements made in Σ', is given by equation (61) as

$$A_I' = -(1 - Z^2/c^2)^{1/2} dZ/dt = -dZ/dT' \tag{63}$$

and the acceleration of a point at rest in the accelerated system, according to physical measurements made in Σ, is given by equation (61) as

$$a_a = (1 - Z^2/c^2)^{1/2} dZ/dT' = dZ/dt \tag{64}$$

The inverse velocity and acceleration transformations may be obtained from equations (57), (58), (61), and (62) by interchanging the primed and unprimed variables and replacing Z by $-Z$. Also, from equations (59) and (60) it can be seen that the accelerated system has a physical boundary at $x' = W^{-1}$.

4. CONCLUSIONS

It is a well-known result of relativity theory that it is impossible to accelerate a perfectly rigid body. It is therefore to be expected that all points in an accelerating frame of reference moving parallel to the x axis of an inertial frame will have instantaneous velocities, as measured in the inertial frame, which vary according to the magnitudes of their x' coordinates within the accelerated frame. The magnitude of this velocity, $Z(x')$, is given by equation (43).

The coordinate transformations between all points in the inertial system and all points in the accelerated system are given by equations (40), which are the only set of equations which satisfy the following criteria:

(1) They give the correct classical equations when $v \ll c$.

(2) In differentiated form they reduce to give the transformation equations of special relativity when $a = 0$ and also when $x' = 0$.

(3) They give the accepted expression for the proper time at $x' = 0$, i.e.,

$$t' = \int_0^t \left(1 - v^2/c^2\right)^{1/2} dt$$

(4) They reduce to give the transformation equations of special relativity when v is a constant.

(5) Physical local measurements of elements of space and time when made at any arbitrary point within the accelerated system are related by the Lorentz transformations to measurements made in the inertial frame. Equations (2)–(5) ensure that this is true for the set of transformations given in equations (40) and in doing so define an accelerated frame of reference Σ' as a frame of reference whose x' axis is moving parallel to the x axis of an inertial frame with velocity $Z(x')$. The magnitude of Z is given by equation (43).

(6) Any observer within the accelerating system must find that the geometry within his immediate infinitesimally small locality is Euclidean. This condition is automatically fulfilled when condition (5) is satisfied.

Conditions (1)–(6) above must be met by any set of transformations between accelerated and inertial frames of reference. The transformations of equation (40) are the only set of equations which obey the above set of conditions while maintaining a linear relationship between x' and x and between t' and x. However, these are only *coordinate transformation equations* and the coordinates x', t' do not correspond to *physical measurements* at points other than the origin of the accelerated system. On the other hand, if real, physical measurements are made by an observer who is fixed at any point in the accelerated system then any measurements he makes of distance and time within his own infinitesimally small locality must yield the same values as those obtained by assuming that a Lorentz frame is instantaneously coincident with, and traveling at the same velocity as, this observer, i.e., the instantaneous relationship between infinitesimal length and time measurements made by this accelerated observer and those made by an inertial observer must be given by the Lorentz transformations with the velocity set equal to Z, i.e., equations (2)–(5). The link between the coordinates of each and every one of such possible observers within the accelerated system is given by the metric of equation (45), and the coordinate points within this metric are related to the coordinate points in the inertial frame by means of equation (40). The fundamental equations (2)–(5) and (40)–(46), therefore, not only provide a set of transformation equations between an inertial frame of reference and an accelerated frame of reference—they also define the metric of the accelerated system and unambiguously define the physical nature of the accelerated frame of reference. The accelerated frame of reference has a physical boundary at $x' = W^{-1}$, which means that when $v = c$, $x' = 0$. In other words, it is impossible to further accelerate any body of finite size when its velocity relative to any given inertial frame is already equal to the velocity of light.

The metric of equation (45) can readily be shown to be identical to the local coordinate system of an accelerated observer as given by Misner, Thorne, and Wheeler (1973) when $v = 0$ and $Wx' \ll 1$. This is to be expected since their coordinates are relative to a comoving inertial frame in which the accelerated observer is momentarily at rest.

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